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Nonlinear convection in a layer with nearly insulating boundaries

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A general class of solutions is studied describing three-dimensional steady convection flows in a fluid layer heated from below with boundaries of low thermal conductivity. Non-linear properties of the solutions are analysed and the physically realizable convection flow is determined by a stability analysis with respect to arbitrary three-dimensional disturbances. The most surprising result is that square-pattern convection is preferred in contrast to two-dimensional rolls that represent the only form of stable convection in a symmetric layer with highly conducting boundaries. The analysis is carried out in the limit of infinite Prandtl number and for a particular boundary configuration. But it is shown that the results hold for arbitrary Prandtl number to the order to which they have been derived and that other assumptions about the boundaries require only minor modifications as long as their thermal conductance remains low.

1. Introduction

The problem of convection in a horizontal layer heated from below has traditionally been considered under the assumption that the boundaries are infinitely conducting such that the deviation θ of the temperature distribution from the static solution vanishes at the boundaries. This boundary condition is often well approximated in laboratory experiments and measurements show good agreement with the theoretical predictions. For recent reviews of linear and nonlinear aspects of convection in a layer heated from below, we refer to the articles by Normand, Pomeau & Velarde (1977) and by Busse (1978). But many geophysical and astrophysical convection problems, as well as those appearing in some engineering applications, do not exhibit well-conducting boundaries, and the ratio β between the thermal conductivities of the boundary and the fluid must be taken into account as an additional parameter.

The importance of the influence of the ratio β on the critical Rayleigh number for the onset of convection was recognized early by Jeffreys (1926), but a systematic analysis of the problem had to await the work of Sani (1963) and of Sparrow, Goldstein & Jonsson (1964). These latter authors included in their analysis the limit of vanishing β in which case the boundaries become insulating. An interesting result of this limit is that the horizontal wavelength of convection rolls tends to infinity and that the critical Rayleigh number equals the integer numbers 120 and 720 for stress-free and rigid boundaries, respectively. An analytical solution of the problem for $\beta = 0$ which illuminates those results was given by Jakeman (1968), who found simple expressions for the convective velocity and the temperature fields even in the case of the more realistic rigid boundaries.

The fact that a simple analytical solution of the linear problem is available in the limit of vanishing β has provided the mathematical motivation for the present study of the nonlinear properties of convection in this limit. From the physical point of view, the limit $\beta \ll 1$ is also attractive because it contrasts most sharply with the intensely studied case of infinitely conducting boundaries. The goal of the analysis of this paper is to isolate the nonlinear properties of the small β limit, rather than to produce results for an extensive range of parameters or for especially realistic cases. In fact, there has been little experimental activity on this aspect of thermal convection, and observational data are not yet available for a comparison with theoretical predictions. It is hoped that the novel phenomena suggested by the theory will stimulate experimental research in this area.

The mathematical approach of this paper follows the small amplitude perturbation study of the case of infinite β by Schlüter, Lortz & Busse (1965), which will be referred to in this paper as SLB. As in the latter case, the analysis of this paper is confronted with the infinite degeneracy of the linear problem. Because of the isotropy and homogeneity of the convection layer, an infinite number of steady, three-dimensional convection flows exist according to the linear theory. Not all of these solutions represent possible solutions of the nonlinear equations in the limit of small amplitude. Thus, the first part of the analysis deals with the problem of the reduction of the degeneracy by the solvability conditions derived from the nonlinear equations. Although the degeneracy is significantly reduced, it can be shown that there still exists an infinite number of possible steady solutions of the full equations. In order to determine the physically realizable solutions, the stability of all solutions with respect to small disturbances must be tested. This is done in the second part of the paper, in which arbitrary three-dimensional disturbances are superimposed onto the steady solutions, and the growth rates are calculated. While it is not possible to distinguish a single physically realizable steady solution, as in the case considered by SLB, a solution forming a square pattern appears to be preferred among a class of closely related solutions.

In order to elucidate the mathematical structure of the problem, only the simplest case is treated explicitly. An infinite Prandtl number of the fluid is assumed and a special boundary configuration is considered. In the discussion at the end of this paper, it is shown that the results are in fact independent of the Prandtl number and that other assumptions about the boundaries can easily be taken into account. This generality of the problem, together with the fact that the analysis involves only rational numbers, contributes to its mathematical attraction.

2. Mathematical formulation of the problem

We consider a horizontal fluid layer of thickness d which is bounded by two infinite half spaces with the thermal conductivity $\lambda^{(e)}$. In the steady static state, a constant heat flux traverses the system such that the temperatures T_1 and T_2 are attained at the upper and lower boundaries of the fluid. It is assumed that the fluid satisfies the Boussinesq approximation, i.e. the kinematic viscosity ν, the conductivity λ and the specific heat c at constant pressure are constant, and the temperature dependence of the density

$$\rho = \rho_0[1-\alpha(T-T_0)]$$

can be neglected everywhere except in the gravity term. For the mathematical description of the problem, non-dimensional variables are introduced based on the length scale d, the time scale $d^2\rho_0 c/\lambda$, and the temperature scale $(T_2-T_1)R^{-1}$. Thus, the equations of motion for the velocity vector $\mathbf{u}$ and the heat equation for the deviation θ of the temperature from the static temperature distribution can be written in the form

$$P^{-1}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla\pi+\theta\boldsymbol{\lambda}+\nabla^2\mathbf{u}, \tag{2.1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.1b}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\theta = R\mathbf{u}\cdot\boldsymbol{\lambda}+\nabla^2\theta, \tag{2.1c}$$

where $\boldsymbol{\lambda}$ is the unit vector in the direction opposite to gravity, $\boldsymbol{\lambda} = -\mathbf{g}/g$, and the Rayleigh number R and the Prandtl number P are defined by

$$R \equiv \frac{\alpha(T_2-T_1)\,g\,d^3\rho_0 c}{\nu\lambda}, \quad P \equiv \frac{\nu\rho_0 c}{\lambda}.$$

It is convenient to introduce a Cartesian system of co-ordinates with the origin on the centre plane of the layer and with the z co-ordinate in the vertical direction. Assuming rigid boundaries the boundary conditions become

$$\mathbf{u} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{2.2a}$$

$$\left.\begin{aligned}\frac{\partial}{\partial z}\theta &= \beta\frac{\partial\theta^{(e)}}{\partial z}\\ \theta &= \theta^{(e)}\end{aligned}\right\} \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{2.2b}$$

where β denotes the conductivity ratio $\lambda^{(e)}/\lambda$ and $\theta^{(e)}$ describes the deviation from the static temperature distribution in the space $|z| \geqslant \frac{1}{2}$.

To eliminate the constraint of the continuity equation (2.1b), we use the general representation

$$\mathbf{u} = \nabla\times(\nabla\times\boldsymbol{\lambda}v)+\nabla\times\boldsymbol{\lambda}\psi \tag{2.3}$$

for the solenoidal vector field $\mathbf{u}$. In order to simplify the problem we shall restrict our attention to the case of infinite Prandtl number, in which case the left-hand side of equation (2.1a) can be neglected. As we shall discuss in § 6, the results remain essentially unchanged when a finite value of P is assumed. By taking the vertical component of the curl of equation (2.1a), it can be shown that the toroidal part $\nabla\times\boldsymbol{\lambda}\psi$ of the velocity field must vanish for $P = \infty$. An equation for the function v is obtained by taking the vertical component of the curl curl of equation (2.1a),

$$\nabla^4\Delta_2 v-\Delta_2\theta = 0 \tag{2.4a}$$

where

$$\Delta_2 \equiv \partial^2/\partial x^2+\partial^2/\partial y^2$$

represents the Laplacian with respect to the horizontal dimensions. In the steady case, the heat equation can be written in the form

$$\nabla^2\theta-R\Delta_2 v = \boldsymbol{\delta}v\cdot\nabla\theta \tag{2.4b}$$

where $\boldsymbol{\delta}v$ stands for $\nabla\times(\nabla\times\boldsymbol{\lambda}v)$. Using the energy method (see, for example, Joseph's 1976 book), it is readily seen that small amplitude steady solutions yield the lowest

Rayleigh number R for which non-decaying solutions exist for equations (2.1) together with boundary conditions (2.2). The assumption of small amplitude steady solutions allows us to drop the right-hand side of equation (2.4*b*) and to obtain a separable solution of the resulting linear problem

$$v_1 = w(x,y)f(z), \tag{2.5a}$$

$$\theta_1 = w(x,y)g(z), \tag{2.5b}$$

where w satisfies the equation

$$\Delta_2 w = -\alpha^2 w. \tag{2.6}$$

(Since the temperature coefficient of the density does not appear in the remainder of the paper, it is expected that it will not be confused with the wavenumber α.) In order to determine the functions $f(z)$ and $g(z)$, the boundary conditions must be expressed in terms of v_1 and θ_1. Using the appropriate solution of

$$\nabla^2\theta^{(e)} = 0 \tag{2.7}$$

with a horizontal dependence given by $w(x,y)$, conditions (2.2) can be written in the form

$$v_1 = \partial v_1/\partial z = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{2.8a}$$

$$\frac{\partial}{\partial z}\theta_1 = \mp\alpha\beta\theta_1 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.8b}$$

From previous work it is known that α vanishes in the limit when β tends to zero. For the investigation of this limit we introduce

$$\gamma \equiv \beta^{\frac{2}{3}} \tag{2.9}$$

as perturbation parameter and anticipate that the wavenumber α_c of the critical convection mode corresponding to the lowest value R becomes proportional to $\beta^{\frac{2}{3}}$ in the limit vanishing γ. Accordingly, it is assumed that the parameter η, defined by the relationship

$$\alpha^2 = \eta^2\gamma, \tag{2.10}$$

is of the order unity for the convection modes of physical interest. The analysis of the following section will demonstrate that the value η_c is indeed independent of γ in the limit $\gamma \to 0$.

Regarding γ as small parameter, the solution v_1, θ_1 of the linear problem can be obtained in terms of a series in powers of γ,

$$v_1 = v_1^{(0)} + \gamma v_1^{(1)} + \gamma^2 v_1^{(2)} + \ldots \tag{2.11}$$

and analogous expressions for θ_1 and R_0. The full problem described by equations (2.4) contains the amplitude ϵ as an additional parameter. Starting with the solution (2.11) of the linear problem, the solution of the full problem can be obtained in terms of a series in powers of ϵ. Thus, the complete solution may be expressed in the form of a double series

$$\left.\begin{aligned} \theta &= \sum_{n=0,\,m=1} \epsilon^m\gamma^n\theta_m^{(n)}, \quad v = \sum_{n=0,\,m=1} \epsilon^m\gamma^n v_m^{(n)}, \\ R &= \sum_{n=0,\,m=0} \epsilon^m\gamma^n R_m^{(n)}. \end{aligned}\right\} \tag{2.12}$$

Because of the nature of the problem, many of the coefficients vanish and only a few must be calculated in order to determine the nonlinear properties of the system in the double limit of small γ and small ϵ.

3. The linear problem

In this section the attention is restricted to the part (2.11) of the general expansion (2.12). Since v_1 enters equation (2.4*b*) only in the order γ, the problem reduces in zeroth order to the equation

$$\frac{\partial^2}{\partial z^2}\theta_1^{(0)} = 0 \tag{3.1a}$$

with the boundary condition

$$\frac{\partial}{\partial z}\theta_1^{(0)} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{3.1b}$$

The obvious solution of problem (3.1) is

$$\theta_1^{(0)} = cw(x,y) \tag{3.2a}$$

where c is a constant. From equation (2.4*a*) we obtain the solution satisfying condition (2.8*a*)

$$v_1^{(0)} = w(x,y)\,c(z^2-\tfrac{1}{4})^2/4\,!. \tag{3.2b}$$

In the order $\epsilon^1\gamma^1$ the equation

$$\frac{\partial^2}{\partial z^2}\theta_1^{(1)} = \eta^2(\theta_1^{(0)} - R_0^{(0)}v_1^{(0)}) \tag{3.3a}$$

is obtained from (2.4*b*). The corresponding boundary condition is

$$\frac{\partial}{\partial z}\theta_1^{(1)} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{3.3b}$$

The coefficient $R_0^{(0)}$ is determined by the condition that a solution of the inhomogeneous boundary-value problem (3.3) exists if and only if the right-hand side of equation (3.3*a*) is orthogonal to all solutions of the adjoint homogeneous problem. By multiplying equation (3.3*a*) by $\theta_1^{(0)}$, partially integrating it and averaging it over the fluid layer, it is readily seen that the operator on the left-hand side of equation (3.3*a*) is self-adjoint and that $R_1^{(0)}$ must satisfy the relationship

$$R_0^{(0)} = \langle\theta_1^{(0)}\theta_1^{(0)}\rangle/\langle\theta_1^{(0)}v_1^{(0)}\rangle \tag{3.4}$$

where the angle brackets indicate the average over the fluid layer. Evaluation of expression (3.4) yields

$$R_0^{(0)} = 720 \tag{3.5}$$

in agreement with the critical Rayleigh number computed in the limit $\beta \to 0$ by Sparrow *et al.* (1964). Equation (3.3*a*) yields the solution

$$\theta_1^{(1)} = \eta^2 c\left(\frac{31}{21\times 2^6} - \tfrac{7}{16}z^2 + \tfrac{5}{4}z^4 - z^6\right)w(x,y) \tag{3.6}$$

where the arbitrary constant of integration has been fixed such that $\theta_1^{(1)}$ is orthogonal to $\theta_1^{(0)}$. For later purposes, it is convenient to introduce as a general normalization condition

$$\langle\theta_n^{(m)}\theta_1^{(0)}\rangle = \delta_{n1}\delta_{m0}. \tag{3.7}$$

By requiring

$$\langle ww\rangle = 1, \tag{3.8}$$

the condition (3.7) allows us to set the as yet undetermined constant c equal to one. Using (3.6), an expression for $v_1^{(1)}$ can be derived from equation (2.4a) and condition (2.8a)

$$v_1^{(1)} = \frac{\eta^2}{7\times 8\times 9\times 10\times 2^{10}}[-(2z)^{10}+15(2z)^8+126(2z)^6 - 810(2z)^4+1187(2z)^2-517]\,w(x,y). \quad (3.9)$$

In order to determine the dependence of the critical Rayleigh number on γ, the order $\epsilon^1\gamma^2$ of equation (2.4b) must be considered:

$$\frac{\partial^2}{\partial z^2}\theta_1^{(2)} = \eta^2(\theta_1^{(1)} - R_0^{(1)}v_1^{(1)} - R_0^{(0)}v_1^{(1)}). \quad (3.10a)$$

The corresponding boundary condition is

$$\frac{\partial}{\partial z}\theta_1^{(2)} = \mp\eta\theta_1^{(0)}. \quad (3.10b)$$

The solvability condition for the inhomogeneous boundary-value problem is obtained in the same way as condition (3.4) and yields

$$\begin{aligned} R_0^{(1)} &= -(R_0^{(0)}\langle v_1^{(1)}\theta_1^{(0)}\rangle - \langle\theta_1^{(0)}\theta_1^{(0)}\rangle\, 2\eta^{-1})\langle v_1^{(0)}\theta_1^{(0)}\rangle^{-1} \\ &= 720\left(\frac{2}{\eta}+\frac{17\eta^2}{6\times 7\times 11}\right). \end{aligned} \quad (3.11)$$

For the onset of convection, the minimum $R_{0c}^{(1)}$ of $R_0^{(1)}$ reached at $\eta = \eta_c$ is of importance,

$$R_{0c}^{(1)} = 2160\left(\frac{17}{6\times 7\times 11}\right)^{\frac{1}{3}}, \quad \eta_c = \left(\frac{6\times 7\times 11}{17}\right)^{\frac{1}{3}}. \quad (3.12)$$

This result demonstrates that the physically relevant value η_c of η is indeed of the order unity and that the expansion scheme (2.11) is internally consistent. At the same time, the result (3.11) suggests that at a small but finite value of β the Rayleigh number tends to infinity as α tends to zero as is expected for finitely conducting boundaries. But this property cannot be firmly concluded from the present analysis because it lies beyond the validity of the perturbation expansion.

4. Nonlinear steady convection

Until this point it has not been necessary to specify the horizontal dependence of the solutions (2.5) beyond the general property (2.6). We shall base the discussion of the manifold of functions $w(x,y)$ on the general solution of equation (2.6)

$$w(x,y) = \sum_{n=-N}^{N} c_n w_n \equiv \sum_{n=-N}^{N} c_n \exp\{i\mathbf{k}_n\cdot\mathbf{r}\} \quad (4.1)$$

where $\mathbf{r}$ is the position vector and where the vectors $\mathbf{k}_n$ satisfy the properties

$$\mathbf{k}_n\cdot\boldsymbol{\lambda} = 0, \quad |k_n| = \alpha, \quad \mathbf{k}_{-n} = -\mathbf{k}_n. \quad (4.2)$$

In order that $w(x,y)$ is a real function which satisfies condition (3.8)

$$\sum_{n=-N}^{N} c_n c_n^+ = 1, \quad c_n^+ = c_{-n}, \quad (4.3)$$

must be required, where c_n^+ denotes the complex conjugate of c_n.

The representation (4.1) is sufficiently general if N is allowed to tend to infinity and if the vectors $\mathbf{k}_n$ cover approximately the full spectrum of horizontal directions. The infinite degeneracy of solution (2.5) expressed by the representation (4.1) is twofold. The fact that a rotation about a vertical axis or a horizontal translation of solution of the form (4.1) leads to a solution of the same form expresses the isotropy and homogeneity of the fluid layer. It cannot be expected that this degeneracy will be affected by the nonlinear properties of the problem. In this section we are concerned about the other degeneracy of the manifold of solutions (2.5) which is expressed by the property that different three-dimensional forms of flows are described by the representation (4.1).

In approaching the hierarchy of equations precipitated by the introduction of the general power series (2.12) into equations (2.4), it becomes evident that

$$v_n^{(m)} \equiv \theta_n^{(m)} \equiv 0 \quad \text{for} \quad n \geqslant m+2, \tag{4.4a}$$

$$R_n^{(m)} = 0 \qquad \text{for} \quad n \geqslant m+2, \tag{4.4b}$$

because the nonlinear term in equation (2.4b) is of the order $\epsilon^2\gamma$. It could be argued that a different expansion parameter might be more appropriate in place of ϵ. Indeed, some of the results derived in this section remain valid for ϵ of the order one. But other results require a small ϵ and thus a change of the perturbation parameter does not seem advantageous.

In the order $\epsilon^2\gamma$ equation (2.4b) yields

$$\frac{\partial^2}{\partial z^2}\theta_2^{(1)} = -\eta^2\Big(R_1^{(0)}v_1^{(0)} + \sum_{n,m} c_n c_m \phi_{nm} w_n w_m z(z^2-\tfrac{1}{4})/3!\Big) \tag{4.5}$$

where ϕ_{nm} is defined by $\qquad \phi_{nm} = \mathbf{k}_n \cdot \mathbf{k}_m/\alpha^2.$

Since the second term on the right-hand side of (4.5) is anti-symmetric with respect to $z = 0$, the solvability condition yields

$$R_1^{(0)} = 0 \tag{4.6}$$

and the solution $\theta_2^{(1)}$ can be written in the form

$$\theta_2^{(1)} = -\frac{\eta^2}{4!}\left(\frac{z^5}{5}-\frac{z^3}{6}+\frac{z}{16}\right)\sum_{\substack{n,m\\ n\neq -m}} c_n c_m \phi_{nm} w_n w_m + \frac{\eta^2}{4!}\left(\frac{z^5}{5}-\frac{z^3}{6}+\frac{7z}{240}\right). \tag{4.7}$$

In deriving this solution, the boundary condition

$$\overline{\theta_2^{(1)}} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{4.8a}$$

has been used for the horizontally averaged component of $\theta_2^{(1)}$ because the horizontal mean of the boundary temperatures is given as an external parameter of the problem. The remaining component of $\theta_2^{(1)}$ satisfies the boundary condition

$$\frac{\partial}{\partial z}\big(\theta_2^{(1)} - \overline{\theta_2^{(1)}}\big) = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{4.8b}$$

Because an explicit expression of $v_2^{(1)}$ is not needed for the later analysis, we proceed directly to the order $\epsilon^3\gamma^2$ of equation (2.4b), which yields

$$\frac{\partial^2}{\partial z^2}\theta_3^{(2)} = -\eta^2 R_2^{(1)} v_1^{(0)} + \boldsymbol{\delta} v_1^{(0)} \cdot \nabla\theta_2^{(1)} + \boldsymbol{\delta} v_2^{(1)} \cdot \nabla\theta_1^{(0)}. \tag{4.9}$$

The solvability condition for this equation requires that the right-hand side is orthogonal to all solutions of the homogeneous problem, i.e. it must vanish after multiplication by w_i^+ and averaging over the fluid layer. Using the z-independence of w_i^+ and $\theta_1^{(0)}$ it is readily seen that

$$\langle w_i^+ \delta v_2^{(1)} . \nabla\theta_1^{(0)} \rangle = \left\langle w_i^+ \left(\frac{\partial}{\partial z}\nabla v_2^{(1)}\right) . \nabla\theta_1^{(0)} \right\rangle = -\left\langle \nabla v_2^{(1)} \frac{\partial}{\partial z}(w_i^+ \nabla\theta_1^{(0)}) \right\rangle = 0. \quad (4.10)$$

After using the relationship

$$\langle w_i^+ \boldsymbol{\delta} v_1^{(0)} . \nabla\theta_2^{(1)} \rangle = -\langle \theta_2^{(1)} \boldsymbol{\delta} v_1^{(0)} . \nabla w_i^+ \rangle$$

the solvability condition yields the set of equations

$$R_2^{(1)} c_i = -A \sum_{\substack{l,m,n \\ n \neq -m}} \phi_{nm} \phi_{li} c_l c_m c_n \langle w_i^+ w_l w_m w_n \rangle + B c_i \quad \text{for} \quad i = -N, \ldots, -1, 1, \ldots, N, \quad (4.11)$$

where A and B are numerical constants given by

$$A = \eta^2 \frac{1}{7 \times 9 \times 2^3}, \quad B = \eta^2 \frac{1}{5 \times 7 \times 3 \times 2^4}.$$

The integral expression in equations (4.11) differs from zero only if

$$-\mathbf{k}_i + \mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n = 0.$$

This condition yields a much simplified set of equations

$$R_2^{(1)} c_i = (B + A \sum_{l \neq i} \phi_{li}^2 |c_l|^2 (2 - \delta_{-li})) c_i \quad \text{for} \quad i = -N, \ldots, -1, 1, \ldots, N, \quad (4.12)$$

where δ_{nm} denotes the Kronecker delta symbol. Equations (4.12), together with condition (4.3), represents an inhomogeneous system of $2N+1$ nonlinear algebraic equations for the $2N$ coefficients c_n and the coefficient $R_2^{(1)}$.

The general solution of equations (4.12) and (4.3) is not known, but a simple set of solutions can be easily derived in the case when the values assumed by ϕ_{li}, $1 \leqslant |l| \leqslant N$, are the same as those assumed by ϕ_{l1}, $1 \leqslant |l| \leqslant N$, for all values of the subscript i. The system of equations is solved in this case by

$$|c_1|^2 = \ldots = |c_N|^2 = 1/2N, \quad (4.13a)$$

$$R_2^{(1)} = B + 2\frac{A}{N} \sum_{l=2}^{N} \phi_{l1}^2 + A/2N. \quad (4.13b)$$

A regular $\mathbf{k}$-vector distribution with the constant angle π/N between neighbouring vectors obviously fulfils the requirement for solution (4.13). But the requirement is also fulfilled in the semi-regular case when each $\mathbf{k}$-vector encloses the angle $2\pi/N$ with each of its second nearest neighbours on either side. As in the problem considered in SLB, the solutions of the form (4.13) for regular and semi-regular $\mathbf{k}$-vector distributions are not further restricted by solvability conditions of higher orders.

The finite value (4.13b) of $R_2^{(1)}$ allows us to express the heat transport H_c by convection in terms of $R-R_c$ for small values of the latter parameter. Using the approximate relationship

$$H_c \approx -\epsilon^2\gamma \frac{\partial}{\partial z}\theta_2^{(1)}\bigg|_{z=\frac{1}{2}}, \tag{4.14}$$

$$R-R_c = \epsilon^2\gamma R_2^{(1)}, \tag{4.15}$$

we find: in the case of two-dimensional solution in the form of rolls,

$$N=1, \quad \frac{H_c^{\text{rolls}}}{R-R_c} = \frac{\eta^2}{4!\,30(B+\frac{1}{2}A)} = \tfrac{7}{8}; \tag{4.16a}$$

in the case of square pattern convection

$$N=2, \quad \phi_{12}=0, \quad \frac{H_c^{\text{squares}}}{R-R_c} = \frac{\eta^2}{4!\,30(B+\frac{1}{4}A)} = \tfrac{14}{11}; \tag{4.16b}$$

and in the case of hexagonal cells

$$N=3, \quad \phi_{12}=\phi_{23}=\phi_{31}=\tfrac{1}{2}, \quad \frac{H_c^{\text{hexagon}}}{R-R_c} = \frac{\eta^2}{4!\,30(B+\frac{1}{2}A)} = \tfrac{7}{8}. \tag{4.16c}$$

The fact that squares exhibit a higher heat transport than rolls suggests a dramatic change in the physical realized pattern of convection from the traditionally considered case $\beta=\infty$. The stability analysis confirms this expectation.

5. Stability analysis

The analysis of the nonlinear equation for steady convection flows has shown that an infinite manifold of solutions exists even though this manifold represents only an infinitesimal fraction of the manifold of solutions (4.1) of the linear problem. To distinguish the physically realizable solution among all possible steady solutions, the stability of v, θ with respect to arbitrary three-dimensional disturbances $\tilde{v}, \tilde{\theta}$ must be investigated. The equations for the time-dependent disturbances are given by

$$\nabla^4\Delta_2\tilde{v} - \Delta_2\tilde{\theta} = 0, \tag{5.1a}$$

$$\nabla^2\tilde{\theta} - R\Delta_2\tilde{v} - \frac{\partial}{\partial t}\tilde{\theta} = \boldsymbol{\delta}v\,.\,\nabla\tilde{\theta} + \boldsymbol{\delta}\tilde{v}\,.\,\nabla\theta. \tag{5.1b}$$

Since the time t does not appear explicitly in the equations, an exponential dependence, $\exp\{\sigma t\}$, can be assumed.

When expansions (2.12) for the steady solution v, θ, R is inserted in equation (5.1b) it becomes evident that equations (5.1) can be solved by an analogous expansion,

$$\tilde{v} = \sum_{n=1,\,m=0} \epsilon^{n-1}\gamma^m\tilde{v}_n^{(m)}, \quad \tilde{\theta} = \sum_{n=1,\,m=0} \epsilon^{n-1}\gamma^m\tilde{\theta}_n^{(m)}, \quad \sigma = \sum_{n=0,\,m=0} \epsilon^n\gamma^m\sigma_n^{(m)}. \tag{5.2}$$

The investigation of the orders $\epsilon^0\gamma^m$ of equations (5.1) yields

$$\sigma_0^{(m)} \leqslant 0 \tag{5.3}$$

if the value η^2 of the steady solution v, θ is chosen such that the Rayleigh number

$$R_0 = \sum_{m=0} \gamma^m R_0^{(m)} \tag{5.4}$$

assumes its minimum value R_c. Since the steady solution itself represents a marginal disturbance as far as linear theory is concerned, it must be expected that the strongest growing disturbances satisfy the same equations as the steady solution in the limit $\epsilon \to 0$. Restricting the attention to the most dangerous disturbances, we assume the equality sign in relationship (5.3) and avoid at the same time the discussion of the more complicated boundary conditions of the time-dependent problem.

Using the representation

$$\tilde{w}(x,y) = \sum_{n=-\infty}^{\infty} \tilde{c}_n w_n \tag{5.5}$$

for the horizontal dependence of the general three-dimensional disturbance, we consider equations (5.1) in higher orders of ϵ. Since relations (4.4) hold for $\tilde{v}_n^{(m)}$, $\tilde{\theta}_n^{(m)}$, $\sigma_n^{(m+1)}$ as well as for $v_n^{(m)}$, $\theta_n^{(m)}$, $R_n^{(m)}$, a non-trivial result can first be obtained in the order $\epsilon^1\gamma^1$ of equation (5.1*b*). But because of the symmetry of the problem with respect to $z = 0$,

$$\sigma_1^{(1)} = 0 \tag{5.6}$$

is found in analogy to (4.6) and solution $\tilde{\theta}_2^{(1)}$ can be written in the form

$$\tilde{\theta}_2^{(1)} = -\frac{\eta_c^2}{4!}\left(\frac{z^5}{5}-\frac{z^3}{6}+\frac{z}{16}\right)\sum_{n=-\infty,\,m=-N}^{n=\infty,\,m=N} 2\tilde{c}_n c_m \phi_{nm} w_n w_m + \frac{\eta_c^2}{4!}\left(\frac{z^5}{5}-\frac{z^3}{6}+\frac{7z}{240}\right) 2\sum_{n=-N}^{N} \tilde{c}_n c_n^{+}. \tag{5.7}$$

The possibility of a non-vanishing positive coefficient $\sigma_n^{(m)}$ appears first in the order $\epsilon^2\gamma^2$, where the solvability condition yields the following set of equations in analogy to (4.12),

$$\begin{aligned}-\sigma_2^{(2)} D\tilde{c}_i = {} & -\underline{R_2^{(1)}\tilde{c}_i} + B\left(\underline{c_i\sum_{e=-N}^{N}|c_e|^2} + c_i\sum_{e=-N}^{N}(c_e\tilde{c}_e^{+} + c_e^{+}\tilde{c}_e)\right)\\ & + A\left(c_i\sum_{e\neq i}\phi_{ei}^2(c_e\tilde{c}_e^{+}+c_e^{+}\tilde{c}_e) + \underline{\tilde{c}_i\sum_{e\neq i}\phi_{ei}^2|c_e|^2}\right.\\ & \left. + c_i\sum_{e\neq i,\,-i}\phi_{ei}^2(c_e\tilde{c}_e^{+}+c_e^{+}\tilde{c}_e) + \underline{\tilde{c}_i\sum_{e\neq i,\,-i}\phi_{ei}^2|c_e|^2}\right) \quad \text{for} \quad -N \leqslant i \leqslant N\end{aligned} \tag{5.8}$$

and

$$-\sigma_2^{(2)} D\tilde{c}_i = -R_2^{(1)}\tilde{c}_i + B\tilde{c}_i\sum_{e=-N}^{N}|c_e|^2 + 2A\tilde{c}_i\sum_{e=-N}^{N}\phi_{ie}|c_e|^2 \quad \text{for} \quad |i| > N, \tag{5.9}$$

where the latter equations are generated by functions w_i^{+} with vectors $\mathbf{k}_i$ different from any vector $\mathbf{k}_n$, $-N \leqslant n \leqslant N$, and where D is a constant given by

$$D = 720/\eta_c^2.$$

Using expression (4.12), it is readily seen that the underlined terms cancel in equations (5.8).

The matrix M_{ij} of the coefficients of $\tilde{c}_j$ in the system of linear homogeneous equations (5.8) is given by

$$M_{ij} = \sigma_2^{(2)} D\delta_{ij} + L_{ij}c_i c_j^{+} \tag{5.10}$$

where L_{ij} is defined by

$$L_{ij} = [2B + (4\phi_{ij}^2 - 3\delta_{ij} - 3\delta_{-ij})A]. \tag{5.11}$$

(The reader is reminded that the summation convention is not used in this paper.) The eigenvalues $\sigma_2^{(2)}$ correspond to the zeros of the determinant of the matrix M_{ij}. This determinant can be evaluated by first dividing the rows and columns of M_{ij}

by c_i and c_j^+, respectively, and by then subtracting the ith row from the $-i$th row, $i = 1, \ldots, M$ and adding the $-j$th column to the jth column, $j = 1, \ldots, N$. The resulting determinant can be written in the form

$$\det(\sigma_2^{(2)} D\delta_{ij}/|c_i|^2) \det(\sigma_2^{(2)} D\delta_{ij}/|c_i|^2 + 2L_{ij}) = 0 \tag{5.12}$$

where the subscripts i, j run from 1 to N only. The first determinant in equation (5.12) yields N eigenvalues $\sigma_2^{(2)} = 0$. The corresponding neutral disturbances include those causing a translation of the steady solution. Since they cannot grow, they are not of interest for the stability problem. In discussing the second determinant we shall restrict the attention to regular or semi-regular steady solutions satisfying relationships (4.13). After writing this determinant as a polynomial in σ,

$$\det(\sigma_2^{(2)} D\delta_{ij}/2N + 2L_{ij}) = \sum_{n=0}^{N} a_n(\sigma_2^{(2)})^n = 0 \tag{5.13}$$

we find that the coefficients a_N and a_{N-1} are positive, while a_{N-2} is negative if

$$F(\phi_{ij}) \equiv \sum_{\substack{i,j=1\\ i>j}}^{N} \{16AB(1-4\phi_{ij}^2) + 4A^2(1-16\phi_{ij}^4)\} < 0. \tag{5.14}$$

Since a polynomial exhibiting a change of sign among its coefficients admits at least one positive root, a steady solution is unstable if condition (5.14) can be satisfied.

Condition (5.14) cannot be satisfied in the case of the roll- and square-solutions because a_{N-2} does not exist in the former and $\phi_{12} = 0$ in the latter case. For the hexagon solution condition (5.14) is not satisfied because $F(\phi_{ij})$ vanishes. Any solution with $N = 2$ is unstable according to condition (5.14) unless

$$|\phi_{12}| < \tfrac{1}{2}, \tag{5.15}$$

i.e. if the angle between the two $\mathbf{k}$ vectors lies between 60° and 120°. All solutions of the form (4.13*a*) with $N \geqslant 4$ are unstable mainly because the number of cases for which condition (5.15) is not fulfilled for two vectors $\mathbf{k}_1, \mathbf{k}_2$ exceeds the number of cases for which it is satisfied. For regular solutions this is readily shown by evaluating $F(\phi_{ij})$. Using $\phi_{i,i+1} = \cos\pi/N$ for $i = 1, \ldots, N$ and employing the formula

$$\sum_{n=1}^{N} (a+nb)\cos\frac{2n\pi}{N} = \tfrac{1}{2}Nb,$$

we find the result

$$F(\phi_{ij}) = [2N - N(N-1)](8AB + 10A^2) \tag{5.16}$$

which clearly demonstrates that condition (5.14) is satisfied for $N \geqslant 4$. The corresponding proof for semi-regular solutions is analogous, but more complex and will not be given here.

To prove stability for any steady solution that does not satisfy condition (5.14), the remaining eigenvalues $\sigma_2^{(2)}$ governed by equations (5.9) must be investigated. The determinant for this system of equations can be written as a product of terms of the form

$$\sigma_2^{(2)} D + A\left\{4\sum_{e=1}^{N}(\phi_{ie}^2 - \phi_{1e}^2)|c_e|^2 + 3|c_1|^2\right\}. \tag{5.17}$$

The condition that expression (5.17) vanishes leads to a maximum positive eigenvalue $\sigma_2^{(2)}$ in the case $N = 1$ when $\mathbf{k}_i$ is chosen such that $\phi_{i1} = 0$. Thus, convection in the form of rolls is unstable to disturbances in the form of rolls oriented at a right angle to the original rolls. This result gives another indication of the preferred role of the square-solution which corresponds to the superposition of two roll solutions at a right angle. Indeed, the evaluation of expression (5.16) for the square solution yields

$$\sigma_2^{(2)} D = -3A|c_i|^2 \tag{5.18}$$

because of the property

$$\phi_{1i}^2 + \phi_{2i}^2 = 1 \quad \text{for all} \quad i.$$

For the hexagon solution, the same expression (5.18) for the eigenvalue $\sigma_2^{(2)}$ is obtained. However, since it is known from the analysis of SLB that the hexagon solution is unstable in the case $\beta = \infty$ owing to a negative value of the coefficient a_{N-2} in expression (5.13), it must be expected that condition (5.14) will be satisfied because of a small contribution of higher order in γ. For the square-solution this is not possible in the limit of small γ since the expression (5.16) is negative definite. Thus, we conclude as the result of the stability analysis that among the solutions of the form (4.13a), only those with $N = 2$ and with property (5.15) are physically realizable. Among the latter solutions, the square-solution is clearly distinguished because of its maximum heat transport and because the most strongly growing disturbances of unstable solutions tend to transform them into the square solution.

6. Discussion

In formulating the mathematical problem (2.4) we have assumed the limit of infinite Prandtl number. This assumption allowed us to drop the variable ψ from our consideration and to neglect the nonlinear term

$$P^{-1}\boldsymbol{\delta} \,.\, (\boldsymbol{\delta} v \,.\, \nabla \boldsymbol{\delta} v) \tag{6.1}$$

in equation (2.4a). An inspection of the analysis shows that the results remain unchanged to the order to which they have been derived if an arbitrary value of the Prandtl number is assumed. As in the analysis of Schlüter *et al.* (1965), the toroidal component $\nabla \times \boldsymbol{\lambda}\psi$ of the velocity field is of the order ϵ^3 or higher and thus cannot enter the analysis described in the preceding section. In contrast to the case $\gamma = \infty$ the term (6.1) does not enter the analysis either. While the function $v_2^{(1)}$ will depend on P, $\theta_2^{(1)}$ remains independent of the Prandtl number. Since $\theta_2^{(1)}$ and $\tilde{\theta}_2^{(1)}$ enter the expression for $R_2^{(1)}$ and $\sigma_2^{(2)}$ but $v_2^{(1)}$ and $\tilde{v}_2^{(1)}$ do not because of relationship (4.10), the results derived in this paper do indeed hold for arbitrary values of the Prandtl number. Only if P tends to zero and becomes comparable to $\gamma^{\frac{1}{2}}$ does this conclusion need to be changed. A more detailed analysis which will not be reported here shows that for $\gamma < 0{\cdot}44\,P^2$ the results of this paper hold qualitatively. When the '$<$' sign in this inequality is replaced by a '$\ll$' sign, the results are quantitatively correct.

In formulating the problem addressed in this paper, we have assumed for simplicity infinite half spaces of constant conductivity above and below the fluid layer. But the results of the analysis do not change significantly when more realistic thermal boundary conditions are assumed as long as the thermal conductivity of the boundary is low compared to that of the fluid. Consider, for example, a fluid layer of depth d bounded by two rigid plates of thickness δd with constant temperatures prescribed on the outer boundaries of the plates. The solution of equation (2.7) is given by

$$\theta^{(e)} = w(x,y)\frac{\sinh\alpha|z\pm(\frac{1}{2}+\delta)|}{\sinh\alpha\delta} \quad \text{for} \quad z\begin{Bmatrix} > \frac{1}{2} \\ < -\frac{1}{2}\end{Bmatrix} \tag{6.2}$$

in this case and yields the boundary condition

$$\left.\begin{aligned}\frac{\partial}{\partial z}\theta_1 &= \mp\beta\theta_1\alpha\coth\alpha\delta \\ &= \mp\beta\theta_1(\delta^{-1}+\alpha^2\delta/3+\dots)\end{aligned}\right\} \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{6.3}$$

instead of condition (2.8b). The appropriate definition of the perturbation parameter is

$$\gamma \equiv \beta/\delta \tag{6.4}$$

in place of definition (2.9). But the rest of the analysis remains unchanged except that η disappears from condition (3.10b) and the expression (3.1) for $R_0^{(1)}$ is replaced by

$$R_0^{(1)} = 720\left(\frac{2}{\eta^2}+\frac{17}{6\times7\times11}\eta^2\right) \tag{6.5a}$$

which leads to somewhat different numerical values of $R_{0c}^{(1)}$ and η_c,

$$R_{0c}^{(1)} = 1440\left(\frac{17}{3\times7\times11}\right)^{\frac{1}{2}}, \quad \eta_c = \left(\frac{6\times7\times22}{17}\right)^{\frac{1}{4}}. \tag{6.5b}$$

In the nonlinear part of the analysis, only the property of the large wavelength of convection is essential and thus the results can be applied to a variety of problems with boundaries of low conductivity of which the configuration considered here is but an example. There appears to be no difficulty in finding experimentally realizable configurations.

We have already mentioned that the square pattern solution is distinguished by a maximum heat transport. That this claim is justified at least among all regular and semi-regular solutions is easily shown by an evaluation of expression (4.13b), which yields

$$R_2^{(1)} = B+A(1-3/2N) \quad \text{for} \quad N \geqslant 3 \tag{6.6a}$$

and

$$R_2^{(1)} = B+A(\tfrac{1}{4}+\cos^2\psi) \quad \text{for} \quad N = 2 \tag{6.6b}$$

where ψ represents the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. The close relationship between stability and maximum heat transport is not unexpected in the small amplitude limit. It has been invoked by Malkus & Veronis (1958) in their first study of small amplitude convection and has been derived under rather general conditions by Busse (1967). The proof of the latter paper actually applies to the problem considered in this paper. It was not used here, since the explicit stability analysis yields more detailed information such as the form of the critical disturbances for the unstable solutions.

In the stability analysis only disturbances with the wavenumber α of the steady solution have been considered because disturbances with a different wavenumber yield negative values of $\sigma_0^{(0)}$ in the case $\alpha = \alpha_c$. In the case $\alpha \neq \alpha_c$ the most strongly growing disturbances have wavenumbers $\tilde{\alpha}$ which do not equal α. The stability analysis can be carried out in analogy to that given by Busse (1971), but it will not be presented here, since the main concern of this paper is the pattern rather than the range of wavenumbers of the realized convection flow. We would like to emphasize, however, that the stability analysis of the case $\alpha \neq \alpha_c$, as well as that of the case $\alpha = \alpha_c$ considered in this paper, requires a small but finite value of β in order that Rayleigh number reaches a minimum at a small but finite value of α_c. In the limit of vanishing β the range of amplitudes ϵ for which the analysis applies also vanishes since the Rayleigh number of modes with the critical wavenumber α_c becomes indistinguishable from the Rayleigh number of their higher harmonics. But since the limit $\beta = 0$ is not physically realistic, it will not be given special consideration.

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